Algebraic geometry 2

Exercise Sheet 6

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Exercise 1. Let K be a field and consider the polynomial ring $K[X_1, X_2, \dots]$ with infinitely many variables X_i, $i \in \mathbb{N}$. Set $A := K[X_1, X_2, \dots]/(X_1^2, X_2^2, \dots)$. Show that $\operatorname{Spec} A$ is a Noetherian topological space but A is not a Noetherian ring.

Exercise 2. (1) Let X be a topological space. Show that X is Noetherian if and only if every open U in X is quasi-compact.
(2) Let X be a Noetherian scheme and $U \subset X$ an open subscheme. Show that U is also a Noetherian scheme.

Exercise 3. For a ring A, denote by A_{red} the quotient ring $A/\operatorname{Nil}(A)$, where $\operatorname{Nil}(A)$ is the nilradical of A.

(1) Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_{\text{red}}$ be the sheaf associated with the presheaf $U \mapsto \mathcal{O}_X(U)_{\text{red}}$. Show that $(X, (\mathcal{O}_X)_{\text{red}})$ is a scheme. Is is called the *reduced scheme* associated to X, and denote it by X_{red}. Show that there is a morphism of schemes $i : X_{\text{red}} \to X$, which is a homeomorphism of the underlying topological spaces.
(2) Let $f : X \to Y$ be a morphism of schemes, where X is reduced. Show that there is a unique morphism $g : X \to Y_{\text{red}}$ such that $f = i \circ g$.

Exercise 4. (Recollection from Commutative Algebra). Let A be a commutative ring, $S \subset A$ a multiplicatively closed subset and $I \subset A$ an ideal of A. Let $\varphi : A \to S^{-1}A$, $a \mapsto \frac{a}{1}$, be the canonical ring homomorphism.

(1) Show that $I^S := \{a \in A \mid sa \in I \text{ for some } s \in S\}$ is an ideal in A (it is called the *saturation* of I).
(2) Recall that $S^{-1}I := \{\frac{i}{s} \mid i \in I, s \in S\}$ is an ideal in $S^{-1}R$. Show that $\varphi^{-1}(S^{-1}I) = I^S$.
(3) Show that there is a one to one correspondence between the ideals I in A with $I = I^S$ and the ideals in $S^{-1}A$.

Remark: We will use this exercise in the proof of Proposition 3.20.